

The Canonical Partition Function For Quons

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Abstract

We calculate the canonical partition function Z_N for a system of N free particles obeying so-called ‘quon’ statistics where q is real and satisfies $|q| < 1$ by using simple counting arguments. We observe that this system is afflicted by the Gibbs paradox and that Z_N is independent of q . We demonstrate that such a system of particles obeys the ideal gas law and that the internal energy U (and hence the specific heat capacity C_V) is identical to that of a system of N free particles obeying Maxwell-Boltzmann statistics.

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I. INTRODUCTION

Since first introduced by Kulish and Rashetikhin [1] in 1981 within the framework of an integrable model in quantum field theory, quantum groups [2–6] have become a topic of great interest to theoretical physicists with many suggested applications (e.g. see [7–12]). The simplest of these structures, $SU_q(2)$, is easily constructed from a pair of commuting operators a_1 and a_2 plus their hermitian adjoints which satisfy the q -boson algebra (first introduced by Biedenharn [13], Macfarlane [14])

$$a_i a_i^\dagger - q^{1/2} a_i^\dagger a_i = q^{-N_i/2} \quad i = 1, 2 \quad (1)$$

where q is a real deforming parameter and N_i is the number operator of the i 'th mode. (We note that the case $q = -1$ is not well defined since the commutator of the step operators J_+^q and J_-^q of $SU_q(2)$ is also badly defined). It is obvious that as $q \rightarrow 1$ the relation (1) goes over into the canonical commutation relation appropriate for the description of bosons.

The relation (1) exists in many equivalent forms; by making the transformation

$$b_i = q^{N_i/4} a_i \quad (2)$$

one arrives at the single-particle form of Greenberg's 'quon' algebra [15] viz.

$$b_i b_i^\dagger - q b_i^\dagger b_i = 1 \quad (3)$$

where different modes commute (a consequence of the commutation of the a_i 's). We have now cast (1) into a form which clearly interpolates between bosons and fermions and is perhaps less cumbersome due to the removal of the number operator factors on the RHS. For the remainder of this work, we will consider the b_i 's to be fundamental objects, not composites of the a_i and N_i operators. It should also be noted that equation (3) originally arose in the work of Arik and Coon [16].

The purpose of this paper will be to investigate the statistical thermodynamics of a system of N free (i.e. the single particle Hamiltonian is assumed to be proportional to the number operator) 'quons' obeying the single-particle relation (3). However, as we shall see

in section II, this system is purely bosonic (complete with Bose-Einstein condensation) with the effects of the deforming parameter only appearing in correlation functions *if the b_i 's are assumed to commute* . We are therefore motivated to depart slightly from (3) to Greenberg's multi-mode 'quon' algebra (also studied in [17] and in an 'anyonic' form in [18])

$$b_i b_j^\dagger - q b_j^\dagger b_i = \delta_{ij} \quad (4)$$

i.e. different modes will be assumed to q -mute rather than commute. (This is, of course, a different physical system to that where the b_i 's are assumed to commute).

We assume the existence of a vacuum state $|0\rangle$ which satisfies

$$b_i |0\rangle = 0 \quad (5)$$

and build up the Fock space by applying monomials in creation operators to the vacuum state. We define the n -particle state $|j_1 j_2 j_3 \dots j_n\rangle$ by

$$b_{j_1}^\dagger b_{j_2}^\dagger b_{j_3}^\dagger \dots b_{j_n}^\dagger |0\rangle = |j_1 j_2 j_3 \dots j_n\rangle \quad (6)$$

for all values of q . Fivel [19] and Zagier [20] showed that the resulting Hilbert space is positive definite if $|q| \leq 1$ - for larger values of $|q|$ states with negative squared norm arise and the probability interpretation of quantum mechanics is lost. We therefore will restrict ourselves to $|q| < 1$. We note that in this case, the n -particle state has no definite symmetry properties under particle interchange - this is a consequence of the simple observation that the relation

$$b_i b_j = q b_j b_i \quad (7)$$

can not be consistently imposed unless $q = \pm 1$. (This is easily seen upon interchanging the indices in (7).)

The structure of this paper is as follows. In section two, we derive the canonical partition function Z_N for N free particles quantized according to the relation (4). We show that Z_N

is independent of q and is therefore equal to the expression given by Greenberg [21] for the special case $q = 0$ (“infinite statistics”). In section three, we examine some of the thermodynamic properties of this system and observe that it is afflicted by Gibbs paradox. We also observe this effect in the grand canonical formalism. Concluding remarks are given in section four.

II. THE CANONICAL PARTITION FUNCTION

We begin the derivation by reviewing Greenberg’s calculation of the canonical partition function for N free particles obeying “infinite statistics” i.e. they obey the relation

$$b_i b_j^\dagger = \delta_{ij} \quad (8)$$

which is clearly a special case of the quonic algebra (4). An advantage of studying this system in detail is that it is generally much simpler than general $|q| < 1$, although often having the same qualitative results, as was again noted by Greenberg. We observe that the n -particle states are orthogonal since it is clear that

$$\langle j_1 j_2 \dots j_n | k_1 k_2 \dots k_n \rangle = \delta_{j_1 k_1} \delta_{j_2 k_2} \dots \delta_{j_n k_n} \quad (9)$$

where the last equation follows directly from the defining relation (8). The computation of Z_N is now possible; we proceed as in the Maxwell-Boltzmann case (for example, see [22]). We assume that we are given a set of occupation numbers $\{n_i\}$ which satisfy

$$N = \sum_i n_i \quad (10)$$

It is then simple to see that the number of orthogonal quantum states $g(\{n_i\})$ is given by

$$g(\{n_i\}) = \frac{N!}{\prod_i (n_i!)} \quad (11)$$

We now relate the partition function Z_N to the single particle partition function Z_1 - however Z_1 is unchanged from the normal bosonic case and is therefore given by the expression

$$Z_1 = \frac{V}{\lambda^3} \quad (12)$$

which is easily obtained by standard methods, and we have introduced the mean thermal wavelength of the particles λ , where λ is given by

$$\lambda = \left(\frac{2\pi}{mT} \right)^{\frac{1}{2}}. \quad (13)$$

(We use units throughout such that $\hbar = k_B = 1$). The calculation is completed by noting that the number of orthogonal quantum states is the result obtained for the Maxwell-Boltzmann distribution except for the Gibbs $N!^{-1}$ factor; hence Z_N for particles quantized according to (8) is

$$Z_N = \left(\frac{V}{\lambda^3} \right)^N \quad (14)$$

We will now reason that Z_N is q -independent for $|q| < 1$ - this was proved recently by Werner [23], although we shall obtain this result through simple counting arguments. The critical observation is that all of the vectors in our positive definite Hilbert space are linearly independent, and are orthogonal for the special case $q = 0$. However for general $|q| < 1$, we may imagine a change of basis to an orthogonal set of states $|j_1' j_2' j_3' \dots j_n' \rangle_o$ so that

$$|j_1 j_2 j_3 \dots j_n \rangle = \sum_{j_1' j_2' j_3' \dots j_n'} c_{j_1' j_2' j_3' \dots j_n'} |j_1' j_2' j_3' \dots j_n' \rangle_o \quad (15)$$

This in many ways resembles Greenberg's use of $q = 0$ operators as building blocks for general $|q| < 1$ - see section V of [15] for a complete discussion.

Now we have a similar problem to that at the start of this section in that we have a collection of orthogonal n -particle states. With the aid of the remark that we will require p orthogonal vectors to span the same vector space defined by p linearly independent vectors, it should become clear that $g(\{n_i\})$ is given by (11) for all $|q| < 1$. This in turn ensures that Z_N is independent of q within this range.

We also hope it is now apparent why the alternative physical system described with commuting b_i 's is trivial. It is clear from the above discussion that the canonical partition

function of a free system is determined entirely by the number of orthogonal quantum states - if this is unchanged from the standard Bose case then the resulting Z_N will also be unchanged from the Bose case. Hence, condensation will occur *at the same temperature as in the undeformed i.e. $q = 1$ case*. The only effect of the deformation parameter will be in the correlation functions [24], for example

$$\langle b_i^\dagger b_i \rangle = \frac{1}{e^{\beta\omega_i} - q} \quad (16)$$

where $\langle \dots \rangle$ indicates the thermal average, and ω_i is the constant of proportionality in the single particle Hamiltonian. (This result, along with other thermal averages, is derived in the appendix).

III. QUON STATISTICAL MECHANICS

Armed with the canonical partition function as found in section II, we may now investigate some of the physical properties of the free ‘quon’ gas. The starting point will be the Helmholtz free energy, A , of the system. This is easily expressed as (where we have used (13))

$$A = NT(3\ln\lambda - \ln V) \quad (17)$$

Immediately from the above expression we obtain relations for the pressure P , entropy S and the chemical potential μ via

$$P = -\left(\frac{\partial A}{\partial V}\right) = \frac{NT}{V} \quad (18)$$

$$S = -\left(\frac{\partial A}{\partial T}\right) = \frac{3}{2}N - \frac{A}{T} \quad (19)$$

$$\mu = \left(\frac{\partial A}{\partial N}\right) = \frac{A}{N} \quad (20)$$

It is perhaps worth making comments about all three of the results. Firstly, it is interesting to observe that the free ‘quon’ gas obeys the ideal gas law. Secondly, we note that as

$T \rightarrow 0 (+\infty)$, the entropy $S \rightarrow -\infty (+\infty)$. Finally, we note that there exists a temperature T_c where the chemical potential vanishes, where T_c is given by

$$T_c = \frac{2\pi}{mV^{\frac{2}{3}}} \quad (21)$$

Using the quantities we have derived above, we can now also construct expressions for the total energy of the system, U , and the specific heat capacity per unit volume C_V via

$$U = A + TS = \frac{3}{2}NT \quad (22)$$

$$C_V = \left(\frac{\partial U}{\partial T} \right) = \frac{3}{2}N \quad (23)$$

We note that the results (22) and (23) are identical to those obtained for a Maxwell-Boltzmann gas consisting of N free particles.

It is necessary to view all of these results with a cautious eye, however. Consider for example the Helmholtz free energy of the system A for a fixed density $\rho = N/V$. By definition A must be an extensive quantity i.e. proportional to the size of the system; substituting into (17) we observe that A contains a term which grows like $N \ln N$. This is a well known problem in statistical mechanics and is known as Gibbs paradox - it is for precisely this reason that the Gibbs $N!^{-1}$ correction term was introduced. However, in our case we can not introduce an *ad hoc* correction term; the weightings have been determined by quantum statistical mechanics. As yet, we have only considered the canonical system. As may be anticipated though, Gibbs paradox is also evident in the grand canonical formalism. The grand canonical partition function, \mathcal{Z} , is defined as

$$\mathcal{Z} = \sum_{N=0}^{\infty} z^N Z_N \quad (24)$$

where $z = \exp(\beta\mu)$ is the fugacity. The average particle number, $\langle N \rangle$, is then given by

$$\langle N \rangle = z \frac{\partial(\ln \mathcal{Z})}{\partial z} \quad (25)$$

$$= \frac{zV}{\lambda^3 - zV} \quad (26)$$

where we have used our earlier result for Z_N . Here the problem is that $\langle N \rangle$ should be an extensive variable of the system; however it is manifestly not proportional to the volume V . We also note that as $T \rightarrow T_c$, the average particle number $\langle N \rangle$ diverges since $z \rightarrow 1$ and $\lambda^3 \rightarrow V$. (This is an equivalent statement to Werners assertion that there exists a volume V at a given temperature for which $\langle N \rangle$ diverges).

IV. DISCUSSION

By considering the number of orthogonal quantum states $g(\{n_i\})$ for a given set of occupation numbers $\{n_i\}$, we have calculated the canonical partition function for N free ‘quons’ and shown it to be independent of q for $|q| < 1$. One may then use this result to show that this system shares many physical characteristics with a system of N free particles obeying Maxwell-Boltzmann statistics; specifically the ideal gas law holds, and the total energy of the systems is the same.

We also observe that there exists a (volume dependent) critical temperature T_c at which the chemical potential vanishes. It is found that in the grand canonical formalism the average particle number $\langle N \rangle$ diverges at this critical temperature. Whether or not this is some kind of condensation phenomena appears to be an open question.

More important perhaps is attempting to determine whether this system is ‘physical’. The appearance of Gibbs paradox due to the form of the function $g(\{n_i\})$ and all the problems associated with the mixing of gases is a severe blow. As Werner notes, the thermodynamic limit just does not make sense for this system.

There appear to be two possible routes out of this troublesome situation. Firstly, one may claim that due to very large interaction times, the system has not yet reached an equilibrium state, as was suggested by Greenberg and Mohapatra [25]; this would then enable cosmological bounds to be placed on q .

The second alternative is perhaps more palatable. As was noted within the introduction, it is not possible to relate $b_i b_j$ to $b_j b_i$ in a consistent way for all i and j . However, it is

possible to impose

$$b_i b_j = f(q) b_j b_i \quad i > j \quad (27)$$

$$b_i b_j = \frac{1}{f(q)} b_j b_i \quad i < j \quad (28)$$

where an obvious candidate is $f(q) = q$ although other choices are possible. Effectively, this alters the number of orthogonal quantum states, thus leading us away from Gibbs paradox. We hope to study this possibility in a future work.

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APPENDIX: CALCULATION OF THERMAL AVERAGES

In section III, we quoted the result for the thermal average of the quantity $b_i^\dagger b_i$. For the sake of completeness we derive that expression here.

Since we know both the form of the one-particle energy spectrum (due to us having chosen the free Hamiltonian) and that our single-particle Fock space consists of states $|0\rangle, |1\rangle, |2\rangle, \dots$ we may compute the thermal average of any operator \hat{O} using

$$\langle \hat{O} \rangle = \frac{\sum_{n=0}^{\infty} \langle n | \hat{O} e^{-\beta H} | n \rangle}{\sum_{n=0}^{\infty} \langle n | e^{-\beta H} | n \rangle} \quad (A1)$$

The simplest thermal average to compute (and perhaps the biggest source of confusion) is that of the number operator N . Using the above definition, we find that

$$\langle N \rangle = \frac{\sum_{n=0}^{\infty} n e^{-\beta n \omega}}{\sum_{n=0}^{\infty} e^{-\beta n \omega}} = \frac{1}{e^{\beta \omega} - 1} \quad (A2)$$

i.e. a result which is **unchanged from the bose case**. This implies that (as correctly stated by Vokos and Zachos [24]) q -deformed black-body spectra (e.g. [26]) are mythical.

However, it does not imply that ‘quon’ thermodynamics is the same as bose thermodynamics since the definite symmetry under particle underchange of the bose case has been removed.

We now compute the thermal average given in section III. Using the defining relation (3), it is easy to show that

$$b_i^\dagger b_i |n\rangle = [n]_q |n\rangle \quad b_i b_i^\dagger |n\rangle = [n+1]_q |n\rangle \quad (\text{A3})$$

where the square bracket function $[x]_q$ is given by

$$[x]_q = \frac{x^q - 1}{x - 1} \quad (\text{A4})$$

Using this, it is a matter of simple arithmetic to obtain

$$\langle b_i^\dagger b_i \rangle = \frac{1}{e^{\beta\omega_i} - q} \quad \langle b_i b_i^\dagger \rangle = \frac{e^{\beta\omega}}{e^{\beta\omega_i} - q} \quad (\text{A5})$$

We observe from (A5) that the ratio of the correlation functions is unchanged from the bose case. However, in q -deformed field theories (since the emission and absorption rates are proportional to $[n+1]_q$ and $[n]_q$ respectively) the emission rates are changed - see [24] for a full discussion.

One should also be aware of the perils of interpreting the quantity $\langle b_i^\dagger b_i \rangle$ as the average number of particles in the i 'th energy level [27]. This is (obviously) only correct when the number operator is given by $b_i^\dagger b_i$ i.e. when $q = \pm 1$.

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